Problems 1

1. Consider a single particle in a ring. The position of the particle corresponds to an angle, θ, which varies from 0 to 2π. The states of this particle are functions of θ over the interval, $(0, 2\pi)$. In addition because the ring extends back on itself - i.e., going beyond $\theta = 2\pi$ corresponds to θ returning back to 0 - and the wavefunctions (states of the system) must be continuous, we must have

$$\psi(0) = \psi(2\pi).$$

This is called periodic boundary conditions. The Hamiltonian for this system takes the form,

$$H = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}.$$

There is one angular degree of freedom, θ. The Hamiltonian (there is only kinetic energy) in classical mechanics is

$$H = \frac{L^2}{2mR^2},$$

where L is the angular momentum and mR^2 is the moment of inertia of the particle about the center of the ring. The classical Hamiltonian becomes the quantum Hamiltonian operator when we insert the angular momentum operator,

$$\hat{L} = -i\hbar\frac{d}{d\theta}.$$

The states,

$$\psi_{c1}(\theta) = \frac{1}{\sqrt{\pi}}\cos(\theta)$$

and

$$\psi_{s2}(\theta) = \frac{1}{\sqrt{\pi}}\sin(2\theta)$$

are energy eigenstates - i.e., eigenfunctions of the Hamiltonian operator.

2. What are the energy eigenvalues associated with ψ_{c1} and ψ_{s2}?

3. Show that ψ_{c1} and ψ_{s2} are orthogonal - i.e.,

$$\begin{aligned}\langle\psi_{c1}|\psi_{s2}\rangle &= \int_0^{2\pi}\psi_{c1}^*(\theta)\psi_{s2}(\theta)\,d\theta \\ &= 0.\end{aligned}$$

4. What is the expectation value of angular momentum for the system in state, ψ_{s2}?

5. Show that ψ_{c1} does not have a well-defined value of angular momentum - i.e., show that $\psi_{c1}(\theta)$ is not an eigenfunction of the angular momentum operator, $\hat{L}$.

6. What is the expectation value of angular momentum for a system in state, ψ_{c1}?

7. Show that

$$\psi_{+1}(\theta) = \frac{1}{\sqrt{2\pi}} \exp(i\theta)$$

has a well-defined value of angular momentum. What is this value?

8. What is the probability that a system in state, ψ_{c1}, has angular momentum, $\hbar$?

9. Show that ψ_{+1} is also an energy eigenstate. What is the associated energy eigenvalue?

Problems 2

1. Consider an electron in a 1D box, in energy eigenstate $\psi_n(x)$.

a. Determine an expression for the probability that the electron is found to be within the interval, $(\frac{2}{5}L, \frac{3}{5}L)$?

b. Evaluate your expression for $n = 1$ and 2.

2. Suppose that the particle in a 1D box is in the state,

$$\psi(x) = Ax(L - x).$$

a. Determine real positive A such that $\psi(x)$ is normalized.

b. Expand $\psi(x)$ as a sum over energy eigenstates - i.e., find the coefficients in the expansion.

c. What is the probability that the energy of the particle in state $\psi(x)$ is measured to be $E_3 = 9\hbar^2\pi^2/(2m)$?

Problems 3

1. Harmonic oscillator problems are simplified by noting that the energy eigenfunctions are functions only of $y = x/\alpha$. Treating the eigenfunctions as functions of y rather than x, leads to a scaled Hamiltonian, and associated energy eigenstates and raising and lowering operators.

$$\hat{H} = \frac{1}{2}\left(-\frac{d^2}{dy^2} + y^2\right) = \hat{a}^\dagger\hat{a} + \frac{1}{2},$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(-\frac{d}{dy} + y\right),$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{d}{dy} + y\right),$$

$$\psi_0(y) = \pi^{-1/2}\exp\left(-\frac{y^2}{2}\right),$$

$$\psi_{\upsilon+1}(y) = \frac{1}{\sqrt{\upsilon+1}}\hat{a}^\dagger\psi_\upsilon(y)$$

and

$$\psi_{\upsilon-1}(y) = \frac{1}{\sqrt{\upsilon}}\hat{a}\psi_\upsilon(y)$$

The (scaled) energy eigenvalues are made explicit in the following TISE:

$$\hat{H}\psi_\upsilon(y) = \left(\upsilon + \frac{1}{2}\right)\psi_\upsilon(y).$$

2. Determine the first and second excited states, $\psi_1(y)$ and $\psi_2(y)$, from the ground state, $\psi_0(y)$, using the raising operator, $\hat{a}^\dagger$.

3. Determine the uncertainty in position, y, and associated momentum, $\hat{p} = -i\hbar d/dy$, for the υ th excited state of the harmonic oscillator. Show that they satisfy the uncertainty principle.

4. Determine the transition matrix element,

$$\langle\psi_{\upsilon+1}|y\psi_\upsilon\rangle$$

for the dipole transition from the υ th to $\upsilon + 1$ th state.

5. Determine the transition matrix element,

$$\langle\psi_{\upsilon+2}|y\psi_\upsilon\rangle$$

for the dipole transition from the υ th to $\upsilon + 2$ th state.

Problems 4

Harmonic oscillator problems are simplified by noting that the energy eigenfunctions are functions only of $y = x/\alpha$. Treating the eigenfunctions as functions of y rather than x, leads to a scaled Hamiltonian, and associated energy eigenstates and raising and lowering operators.

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for the dipole transition from the υ th to $\upsilon + 2$ th state.